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The Generalized Uncertainty Relation and The Entropy of Non-stationary and Slowly Changing Reissner-Nordström Black Hole

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Abstract Applying the generalized uncertainty relation to the thin film brick-wall model, the entropy of Dirac Field in Non-stationary and Slowly Changing Reissner-Nordström Black Hole is obtained. The result shows that the entropy is still proportional to the horizon area of the black hole, and black hole entropy is just identical to the entropy of the quantum state near the event horizon, in addition, the divergence of state density without any cut-off parameter is avoided during black hole entropy calculation.

Keywords Generalized uncertainty relation · State density · Quantum entropy · Dirac equation

1 Introduction

Since Hawking proved, in 1974, black hole can radiate particles from the event horizon, the thermodynamical proprieties of black holes have attracted many peoples' attentions [1–34]. In 1970s, Bekenstein put forward that black hole entropy was proportional to the horizon area [35]. After that, Hawking discovered the thermal radiation of black hole, and proved the Bekenstein's conjecture. Since then, the research on the statistical origin of black hole entropy is an important topic to theoretical physicist. t'Hooft first proposed the brick-wall model to calculate black hole entropy, in which black hole entropy is identified with the entropy of a thermal gas of quantum field excitations outside the event horizon, to avoid the divergence of the state density, they introduce the cut-off parameter [36–39]. In 1995, Demoers found that the divergence appearing in the brick-wall model could be absorbed into the renormalized Newton's constant [40]. From then on, people had a deep research into the relation among the brick-wall model, the quantum gravity and renormalization. In

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recent years, the entropy of a series of black holes have been studied by the improved brick-wall model put forward by Li and Zhao, and much success is achieved [41–47]. But they still kept the cut-off parameter, and made the results unnatural. Recently, as the advance of research into the quantum gravity, people have introduced a revised uncertainty relation to the calculation of black hole entropy, and got the satisfying result. The method avoids the divergence of state density near the horizon without any cut-off parameter. Since then, much effort has been devoted to the entropies of static or stationary black holes [48–50], but those of the non-stationary black hole are limited to the line element that are described by the advanced or retarded coordinate [51, 52]. As to those of the non-stationary space-time described by the coordinate (t, r, θ, φ) are still unfamiliar to us. In the paper, we apply the generalized uncertainty relation to the thin film brick-wall model, and discuss the entropy of Dirac Field in Non-stationary and Slowly Changing Reissner-Nordström Black Hole that is described by the coordinate (t, r, θ, φ).

2 The Separation of Variables

In quantum mechanics, the simplest uncertainty of position x and momentum p should satisfy Heisenberg uncertainty relation

$$\Delta x \Delta p \geq \hbar. \tag{1}$$

However, in the quantum system under the Plank scale like quantum gravitation system, (1) is not correct, and we should introduce the generalized uncertainty relation

$$\Delta x \Delta p \geq \frac{\hbar}{2}[1 + \lambda\langle p^2\rangle + \beta\langle x^2\rangle + \cdots]. \tag{2}$$

Only considering the leading role of momentum (namely, $\beta = 0$, $\lambda \neq 0$), we have

$$\Delta x \Delta p \geq \frac{\hbar}{2}[1 + \lambda(\hat{p}^2)] = \frac{\hbar}{2}[1 + \lambda(\Delta p^2 + \langle\hat{p}\rangle^2)], \tag{3}$$

where $\hbar$ is the Plank constant, λ is of Plank scale. From (3), we can obtain that the position uncertainty Δx cannot be small enough, and Δx has a minimal length $2\sqrt{\lambda}$. According to the general statistical physics, as a result of the uncertainty relation (1), a quantum state occupies a phase element whose volume is $(2\pi\hbar)^3$, so the numbers of the quantum states in the volume of d^3xd^3p are

$$\frac{d^3xd^3p}{(2\pi\hbar)^3}. \tag{4}$$

But if we introduce the generalized uncertainty relation (3), the quantum state numbers in the volume of d^3xd^3p can be written as

$$\frac{d^3xd^3p}{(2\pi\hbar)^3(1 + \lambda p^2)^3}, \tag{5}$$

where in the curved space-time $p^2 = p_i p^i = g^{i\mu} p_i p_\mu$, $g^{i\mu}$ is the inverse metric tensor, $i = 1$, 2, 3; $\mu = 0, 1, 2, 3$. From (5), the new state density is given by [53]

$$g(\omega) = \frac{1}{(2\pi)^3} \int \frac{dr d\theta d\varphi dp_r dp_\theta dp_\varphi}{(1 + \lambda p^2)^3}. \tag{6}$$

According to [54], the line element of the non-stationary and slowly changing Reissner-Nordström black hole in the time coordinate t with the $(+,-,-,-)$ metric sign has the following form

$$ds^2=\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2, \quad (7)$$

where, $M=M(t)$ and $Q=Q(t)$ are the mass and charge of the non-stationary and slowly changing Reissner-Nordström black hole respectively, as seen by an observer at infinity, they are changing with the time coordinate t. Introducing the new tortoise coordinate transformation [55, 56]

$$r_*=\frac{1}{2\kappa}\ln[r-r_H(t)], \quad (8)$$

we can get the space-time line element in two dimensions near the event horizon

$$ds^2=2\kappa(r-r_H)\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}\dot{r}_H\left[\frac{(r^2-2Mr+Q^2)^2-\dot{r}_H^2r^4}{2\kappa(r-r_H)\dot{r}_Hr^4}dt_*^2-2dr_*dt_*\right]. \quad (9)$$

In (9), when $r\to r_H$, the denominator in the coefficient of dt_*^2 is null, and the factor is also null in order to avoid the divergence, so we have

$$\lim_{r\to r_H}[(r^2-2Mr+Q^2)^2-\dot{r}_H^2r^4]=0. \quad (10)$$

Solving (9), we can get the locations of the event horizons

$$r_H=r_+=\frac{M+\sqrt{M^2-Q^2(1-\dot{r}_H)}}{1-\dot{r}_H}, \qquad r_H=r_-=\frac{M-\sqrt{M^2-Q^2(1-\dot{r}_H)}}{1-\dot{r}_H}, \quad (11)$$

where r_+ and r_- are the event horizons of the non-stationary and slowly changing Reissner-Nordström black hole. If we make the space-time line element (9) in the vicinity of the event horizon similar to the Minkowski space-time in two dimensions, the coefficient of dt_*^2 must be equal to 1, so we obtain

$$\lim_{r\to r_H}\frac{(r^2-2Mr+Q^2)^2-\dot{r}_H^2r^4}{2\kappa(r-r_H)\dot{r}_Hr^4}=1. \quad (12)$$

From (12), we can get the surface gravity of the event horizon

$$\kappa=\frac{Mr_H-Q^2}{r_H^3}=\frac{(1-\dot{r}_H)(r_H^2-r_H^2\dot{r}_H-Q^2)}{2r_H(2Mr_H-Q^2)}, \quad (13)$$

and the radiate temperature

$$T=\frac{\kappa}{2\pi}=\frac{Mr_H-Q^2}{\pi r_H^3}=\frac{(1-\dot{r}_H)(r_H^2-r_H^2\dot{r}_H-Q^2)}{2\pi r_H(2Mr_H-Q^2)}. \quad (14)$$

For the inverse metric $g^{\mu\nu}$ of (7), we obtain

$$\frac{\partial^2}{\partial s^2}=\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}\frac{\partial^2}{\partial t^2}-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2},$$
$$g=\det g_{\mu\nu}=-r^4\sin^2\theta. \quad (15)$$

To write out the explicit form of Dirac equation in the Newman-Penrose formalism, we establish the null tetrads as follows

$$\begin{aligned} l_\mu &= \frac{B}{2r^2}\left(1, -\frac{r^2}{B}, 0, 0\right), \qquad n_\mu = \left(1, \frac{r^2}{B}, 0, 0\right), \\ m_\mu &= \frac{r}{\sqrt{2}}(0, 0, -1, -i\sin\theta), \qquad \bar{m}_\mu = \frac{r}{\sqrt{2}}(0, 0, -1, i\sin\theta). \end{aligned} \tag{16}$$

From (16), we can get the inverse null tetrads

$$z_m^\mu = (z_1^\mu, z_2^\mu, z_3^\mu, z_4^\mu) = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu), \tag{17}$$

where

$$\begin{aligned} l^\mu &= \frac{1}{2}\left(1, \frac{B}{r^2}, 0, 0\right), \qquad n^\mu = \left(\frac{r^2}{B}, -1, 0, 0\right), \\ m^\mu &= \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{i}{\sin\theta}\right), \qquad \bar{m}^\mu = \frac{1}{\sqrt{2}r}\left(0, 0, 1, -\frac{i}{\sin\theta}\right), \end{aligned} \tag{18}$$

where $B = r^2 - 2Mr + Q^2$. The null tetrads (16), (18) satisfy the null, orthogonal and metric conditions.

$$\begin{aligned} & l^\mu l_\mu = n^\mu n_\mu = m^\mu m_\mu = \bar{m}^\mu \bar{m}_\mu = 0, \qquad l^\mu m_\mu = l^\mu \bar{m}_\mu = n^\mu m_\mu = n^\mu \bar{m}_\mu = 0, \\ & l^\mu n_\mu = -m^\mu \bar{m}_\mu = 1, \quad \text{and} \quad g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu. \end{aligned} \tag{19}$$

From the inverse null tetrad (18), we can obtain the corresponding directional derivatives of the space-time

$$\begin{aligned} D &= l^\mu \partial_\mu = \frac{1}{2}\frac{\partial}{\partial t} + \frac{B}{2r^2}\frac{\partial}{\partial r}, \qquad \Delta = n^\mu \partial_\mu = \frac{r^2}{B}\frac{\partial}{\partial t} - \frac{\partial}{\partial r}, \\ \delta &= m^\mu \partial_\mu = \frac{1}{\sqrt{2}r}\frac{\partial}{\partial\theta} + \frac{i}{\sqrt{2}r\sin\theta}\frac{\partial}{\partial\varphi}, \qquad \bar{\delta} = \bar{m}^\mu \partial_\mu = \frac{1}{\sqrt{2}r}\frac{\partial}{\partial\theta} - \frac{i}{\sqrt{2}r\sin\theta}\frac{\partial}{\partial\varphi}. \end{aligned} \tag{20}$$

According to (7), (15) and $\Gamma^\alpha_{\mu\sigma} = \frac{1}{2}g^{\alpha\nu}(g_{\mu\nu,\sigma} + g_{\nu\sigma,\mu} - g_{\sigma\mu,\nu})$, after some complicate calculation, the non-vanishing spin coefficients of the non-stationary and slowly changing Reissner-Nordström black hole in the above null tetrads (16) and (18) can be written as

$$\begin{aligned} & \varepsilon = \frac{Mr - Q^2}{2r^3}, \qquad \tilde{\rho} = -\frac{B}{2r^3}, \qquad \beta = -\alpha = \frac{\mathrm{ctg}\,\theta}{2\sqrt{2}r}, \\ & \mu = -\frac{1}{r}, \qquad \gamma = \frac{r^2}{B^2}(\dot{Q}Q - \dot{M}r). \end{aligned} \tag{21}$$

In the curved space-time described by the line element (7), the four components of Dirac equations can be expressed as

$$\gamma^\mu(\partial_\mu\psi - \Gamma_\mu\psi) + i\mu_0\psi = 0, \tag{22}$$

where

$$\Gamma_\mu = -\frac{1}{4}\gamma^\beta(\gamma_{\beta,\mu} - \gamma_\lambda \Gamma^\lambda_{\beta\mu}). \tag{23}$$

In the curved space-time, if we use the spin coefficients suggested by Penrose, Dirac equation of particle with no charges can be written as

$$\begin{aligned}
(D + \varepsilon - \tilde{\rho})F_1 + (\bar{\delta} + \pi - \alpha)F_2 &= \frac{i\mu_0}{\sqrt{2}}G_1,\\
(\Delta + \mu - \gamma)F_2 + (\delta + \beta - \tau)F_1 &= \frac{i\mu_0}{\sqrt{2}}G_2,\\
(D + \varepsilon^* - \tilde{\rho}^*)G_2 - (\delta + \pi^* - \alpha^*)G_1 &= \frac{i\mu_0}{\sqrt{2}}F_2,\\
(\Delta + \mu^* - \gamma^*)G_1 - (\bar{\delta} + \beta^* - \tau^*)G_2 &= \frac{i\mu_0}{\sqrt{2}}F_1,
\end{aligned} \tag{24}$$

where μ_0 is the mass of Dirac particles; F_1, F_2, G_1, G_2 are the four components of wave function in Dirac field. Substituting (20), (21) into (24), we have

$$\begin{aligned}
\left(\frac{1}{2}\frac{\partial}{\partial t} + \frac{B}{2r^2}\frac{\partial}{\partial r} + \frac{r-M}{2r^2}\right)F_1 + \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\mathrm{ctg}\,\theta}{2}\right)F_2 &= \frac{i\mu_0}{\sqrt{2}}G_1,\\
\left[\frac{r^2}{B}\frac{\partial}{\partial t} - \frac{\partial}{\partial r} - \frac{1}{r} - \frac{r^2}{B^2}(\dot{Q}Q - \dot{M}r)\right]F_2 + \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\mathrm{ctg}\,\theta}{2}\right)F_1 &= \frac{i\mu_0}{\sqrt{2}}G_2,\\
\left(\frac{1}{2}\frac{\partial}{\partial t} + \frac{B}{2r^2}\frac{\partial}{\partial r} + \frac{r-M}{2r^2}\right)G_2 - \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\mathrm{ctg}\,\theta}{2}\right)G_1 &= \frac{i\mu_0}{\sqrt{2}}F_2,\\
\left[\frac{r^2}{B}\frac{\partial}{\partial t} - \frac{\partial}{\partial r} - \frac{1}{r} - \frac{r^2}{B^2}(\dot{Q}Q - \dot{M}r)\right]G_1 - \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\mathrm{ctg}\,\theta}{2}\right)G_2 &= \frac{i\mu_0}{\sqrt{2}}F_1.
\end{aligned} \tag{25}$$

As a result of the space-time sphere-symmetry, we set the four components of wave function as following form

$$\begin{aligned}
F_1 &= f_1(r,t)Y_1(\theta,\varphi), \qquad & F_2 &= f_2(r,t)Y_2(\theta,\varphi),\\
G_1 &= g_1(r,t)Y_1(\theta,\varphi), \qquad & G_2 &= g_2(r,t)Y_2(\theta,\varphi).
\end{aligned} \tag{26}$$

Substituting (26) into (25) can we obtain

$$\begin{aligned}
\left(\frac{1}{2}\frac{\partial}{\partial t} + \frac{B}{2r^2}\frac{\partial}{\partial r} + \frac{r-M}{2r^2}\right)f_1 + \frac{\chi_1}{\sqrt{2}r}f_2 - \frac{i\mu_0}{\sqrt{2}}g_1 &= 0,\\
\left[\frac{r^2}{B}\frac{\partial}{\partial t} - \frac{\partial}{\partial r} - \frac{1}{r} - \frac{r^2}{B^2}(\dot{Q}Q - \dot{M}r)\right]f_2 + \frac{\chi_2}{\sqrt{2}r}f_1 - \frac{i\mu_0}{\sqrt{2}}g_2 &= 0,\\
\left(\frac{1}{2}\frac{\partial}{\partial t} + \frac{B}{2r^2}\frac{\partial}{\partial r} + \frac{r-M}{2r^2}\right)g_2 - \frac{\chi_2}{\sqrt{2}r}g_1 - \frac{i\mu_0}{\sqrt{2}}f_2 &= 0,\\
\left[\frac{r^2}{B}\frac{\partial}{\partial t} - \frac{\partial}{\partial r} - \frac{1}{r} - \frac{r^2}{B^2}(\dot{Q}Q - \dot{M}r)\right]g_1 - \frac{\chi_1}{\sqrt{2}r}g_2 - \frac{i\mu_0}{\sqrt{2}}f_1 &= 0,
\end{aligned} \tag{27}$$

where χ_1 and χ_2 are constants used in the process of variable separation

$$\chi_1 = \frac{1}{Y_1}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\operatorname{ctg}\theta}{2}\right)Y_2, \qquad \chi_2 = \frac{1}{Y_2}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\operatorname{ctg}\theta}{2}\right)Y_1. \tag{28}$$

An apparent fact is that the Chandrasekhar-Dirac equation (27) could be satisfied by settings

$$f_1 = g_2, \qquad f_2 = g_1, \qquad \chi_1 = -\chi_2 = \chi, \tag{29}$$

obviously contrasting to each item in (27), we will find that (29) is rational. Substituting (29) into (27), we can obtain Dirac equations in radial

$$\begin{aligned} &\left(\frac{1}{2}\frac{\partial}{\partial t} + \frac{B}{2r^2}\frac{\partial}{\partial r} + \frac{r-M}{2r^2}\right)f_1 + \frac{\chi}{\sqrt{2}r}f_2 - \frac{i\mu_0}{\sqrt{2}}f_2 = 0, \\ &\left[-\frac{r^2}{B}\frac{\partial}{\partial t} + \frac{\partial}{\partial r} - \frac{1}{r} - \frac{r^2}{B^2}(\dot{Q}Q - \dot{M}r)\right]f_2 - \frac{\chi}{\sqrt{2}r}f_1 - \frac{i\mu_0}{\sqrt{2}}f_1 = 0, \end{aligned} \tag{30}$$

and in angular

$$\hat{L}_- Y_1 = -\chi^2 Y_1, \qquad \hat{L}_+ Y_2 = -\chi^2 Y_2, \tag{31}$$

where

$$\hat{L}_\mp = \frac{\partial^2}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\left(\frac{\cos^2\theta}{4} \mp i\cos\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\varphi^2} - \frac{1}{2}\right). \tag{32}$$

The constant of variable separation χ satisfies

$$\chi^2 = (l + s_z)(l + s_z + 1), \tag{33}$$

where $l > \frac{1}{2}$, $s_z = \pm\frac{1}{2}$ is the spin quantum states, and it denotes that every point in the phase space corresponds to two states.

3 The Black Hole Entropy

In radial equation (30), a rational approximate solution can be expressed by the field function $\phi \approx \exp[-i\omega t + iS(r,\theta,\varphi)]$, and applying the WKB approximation, we can obtain

$$\begin{aligned} &\left[-\frac{i\omega}{2} + \frac{iB}{2r^2}\frac{\partial S}{\partial r}\right]f_1 - \left(\frac{i\mu_0}{\sqrt{2}} - \frac{\chi}{\sqrt{2}r}\right)f_2 = 0, \\ &\left[-\frac{i\omega r^2}{B} - \frac{i\partial S}{\partial r}\right]f_2 - \left(\frac{i\mu_0}{\sqrt{2}} + \frac{\chi}{\sqrt{2}r}\right)f_1 = 0. \end{aligned} \tag{34}$$

If we want to make the solution of (34) meaningful, the determinant of the coefficient matrix must be null, so we can obtain

$$p_r^2 = \left(\frac{\partial S}{\partial r}\right)^2 = \frac{r^2}{B}\left(\frac{r^2}{B}\omega^2 - \mu_0^2 - \frac{\chi^2}{r^2}\right). \tag{35}$$

Now let us solve the red-shift factor of wave length in r-direction

$$(dr)^2 = -\gamma_{11}(dr')^2, \qquad \gamma_{11} = g_{11} - \frac{g_{01}^2}{g_{00}}, \tag{36}$$

namely

$$\lambda = \sqrt{-\gamma_{11}}\lambda' = \frac{1}{\sqrt{g_{00}}}\lambda'. \tag{37}$$

From (37), we can obtain

$$p^2 = \omega'^2 = \left(\frac{2\pi}{\lambda'}\right)^2 = \frac{\omega^2}{g_{00}} = \frac{r^2\omega^2}{B}. \tag{38}$$

According to [57], and substituting (35), (38) into the new state density (6), we can obtain the numbers of quantum states in the two spin quantum states with the spins $s_z = \frac{1}{2}$ and $s_z = -\frac{1}{2}$.

$$\begin{aligned} g_1(\omega) &= \frac{2}{3\pi}\int_{r_H}^{r_H+\varepsilon} \frac{r^2}{(B/r^2)^2(1+\lambda\omega^2 r^2/B)^3}\left(\omega^2 - \frac{B}{r^2}\mu_0^2\right)^{3/2} dr, \qquad s_z = \frac{1}{2}, \\ g_1(\omega) &= \frac{2}{3\pi}\int_{r_H}^{r_H+\varepsilon} \frac{r^2}{(B/r^2)^2(1+\lambda\omega^2 r^2/B)^3}\left(\omega^2 - \frac{B}{r^2}\mu_0^2 - \frac{B}{r^4}\right)^{3/2} dr, \qquad s_z = -\frac{1}{2}. \end{aligned} \tag{39}$$

Considering the null mass approximation and the big scale space-time background of black hole in (39), the two spin states should have the same numbers of quantum states. So the total number of quantum states is

$$g(\omega) \approx \frac{4\omega^3}{3\pi}\int_{r_H}^{r_H+\varepsilon} \frac{r^2 dr}{(B/r^2)^2(1+\lambda\omega^2 r^2/B)^3}. \tag{40}$$

According to the quantum statistics theory, the free energy of Dirac particles can be expressed

$$\begin{aligned} F &= \frac{1}{\beta}\int dg(\omega)\ln(1+e^{-\beta\omega}) = -\int_0^\infty \frac{1}{(e^{\beta\omega}+1)} g(\omega) d\omega \\ &= -\frac{4}{3\pi}\int_{r_H}^{r_H+\varepsilon} \frac{r^2 dr}{(B/r^2)^2}\int_0^\infty \frac{\omega^3 d\omega}{(e^{\beta\omega}+1)(1+\lambda\omega^2 r^2/B)^3}. \end{aligned} \tag{41}$$

Because the relation between the free energy and the entropy of the black hole satisfies

$$\begin{aligned} S &= \beta^2\frac{\partial F}{\partial\beta} = \frac{4\beta^2}{3\pi}\int_{r_H}^{r_H+\varepsilon} \frac{r^2 dr}{(B/r^2)^2}\int_0^\infty \frac{e^{\beta\omega}\omega^4 d\omega}{(e^{\beta\omega}+1)^2(1+\lambda\omega^2 r^2/B)^3} \\ &= \frac{4}{3\pi}\int_{r_H}^{r_H+\varepsilon} \frac{r^2 dr}{\beta^3(B/r^2)^2}\int_0^\infty \frac{x^4 dx}{(e^{-x}+1)(e^x+1)(1+\lambda x^2 r^2/\beta^2 B)^3}, \end{aligned} \tag{42}$$

where $x = \beta\omega$, and the inequality

$$\frac{1}{1+e^{-x}} < e^x, \qquad \frac{1}{1+e^x} < e^{-x}, \tag{43}$$

we have

$$
\begin{aligned}
S &< \frac{4}{3\pi}\int_{r_H}^{r_H+\varepsilon}\frac{r^2dr}{\beta^3(B/r^2)^2}\int_0^\infty \frac{x^4dx}{(1+\lambda x^2r^2/\beta^2B)^3}\\
&= \frac{4}{3\pi}\int_{r_H}^{r_H+\varepsilon}\frac{r^2dr}{\beta^3(B/r^2)^2}\left[\frac{1}{4}\left(\frac{\lambda r^2}{\beta^2 B}\right)^{-2}+\frac{\pi}{16}\left(\frac{\lambda r^2}{\beta^2 B}\right)^{-\frac{3}{2}}\right]\\
&= \frac{\beta}{3\pi\lambda^2}\int_{r_H}^{r_H+\varepsilon}r^2dr+\frac{\lambda^{-3/2}}{12}\int_{r_H}^{r_H+\varepsilon}\frac{r^2dr}{\sqrt{B/r^2}}.
\end{aligned}
\tag{44}
$$

Using Taylor series to launch $\frac{B}{r^2}$ at the event horizon, and we have

$$
\frac{B}{r^2}=\left(1-\frac{2M}{r_H}+\frac{Q^2}{r_H^2}\right)+\frac{2Mr_H-2Q^2}{r_H^3}(r-r_H)+\cdots\approx 2\kappa(r-r_H). \tag{45}
$$

The uncertainty of the position has a minimal length $2\sqrt{\lambda}$, so we have the following expression near the event horizon

$$
2\sqrt{\lambda}=\int_{r_H}^{r_H+\varepsilon}\frac{dr}{\sqrt{B/r^2}}\approx\int_{r_H}^{r_H+\varepsilon}\frac{dr}{\sqrt{2\kappa(r-r_H)}}=\sqrt{\frac{2\varepsilon}{\kappa}}, \tag{46}
$$

where $\kappa=2\pi\beta^{-1}$. Substituting (45), (46) into (44), we can get the quantum entropy of the black hole near the event horizon

$$
S_H\propto\frac{\beta}{3\pi\lambda^2}r_H^2\varepsilon+\frac{\lambda^{-3/2}}{6}\sqrt{\lambda}r_H^2=\frac{3A_H}{8\pi\lambda}, \tag{47}
$$

where $A_H=4\pi r_H^2$ is the horizon area of the black hole. From (47), we can learn that the entropy is proportional to the horizon area of the black hole, which is consistent with Bekenstein-Hawking theory. During the calculation, we do not introduce the cut-off parameter, but apply the generalized uncertainty relation to obtain the numbers of new quantum states. This method avoids the divergence of state density at the event horizon, and finally gets the entropy of the black hole.

4 Conclusions

We start from Dirac equation in the Non-stationary and Slowly Changing Reissner–Nordström space-time background, and use the generalized uncertainty relation to obtain the numbers of new quantum states, finally research on the entropy of Dirac field in the black hole. The result shows that the divergence of brick-wall model near the horizon can be eliminated without any cut-off parameter when the generalized uncertainty relation is used to study the entropy of black hole, and the black hole entropy is proportional to the horizon area of the black hole, and is just identical to the entropy of quantum state near the horizon.

References

1. Parikh, M.K., Wilczek, F.: Phys. Rev. Lett. **85**, 5042 (2000)
2. Robinson, S.P., Wilczek, F.: Phys. Rev. Lett. **95**, 011303 (2005)

3. Hemming, S., Keski-Vakkuri, E.: Phys. Rev. D **64**, 044006 (2001)
4. Medved, A.J.M.: Phys. Rev. D **66**, 124009 (2002)
5. Medved, A.J.M.: Class. Quantum Gravity **19**, 589 (2002)
6. Vagenas, E.C.: Phys. Lett. B **503**, 399 (2001)
7. Vagenas, E.C.: Phys. Lett. B **559**, 65 (2003)
8. Vagenas, E.C.: Mod. Phys. Lett. A **17**, 609 (2002)
9. Medved, A.J.M., Vagenas, E.C.: Mod. Phys. Lett. A **20**, 2449 (2005)
10. Medved, A.J.M., Vagenas, E.C.: Mod. Phys. Lett. A **20**, 1723 (2005)
11. Arzano, M., Medved, A.J.M., Vagenas, E.C.: J. High Energy Phys. **0509**, 037 (2005)
12. Iso, S., Umetsu, H., Wilczek, F.: Phys. Rev. Lett. **96**, 151302 (2006)
13. Iso, S., Umetsu, H., Wilczek, F.: Phys. Rev. D **74**, 044017 (2006)
14. Iso, S., Morita, T., Umetsu, H.: J. High Energy Phys. **0704**, 068 (2007)
15. Iso, S., Morita, T., Umetsu, H.: Phys. Rev. D **75**, 124004 (2007)
16. Iso, S., Morita, T., Umetsu, H.: Phys. Rev. D **76**, 064015 (2007)
17. Iso, S., Morita, T., Umetsu, H.: Nucl. Phys. B **799**, 60 (2008)
18. Iso, S., Morita, T., Umetsu, H.: Phys. Rev. D **77**, 045007 (2008)
19. Iso, S.: Int. J. Mod. Phys. A **23**, 2082 (2008)
20. Jiang, Q.Q., Wu, S.Q., Cai, X.: Phys. Rev. D **75**, 064029 (2007)
21. Jiang, Q.Q., Wu, S.Q., Cai, X.: Phys. Rev. D **73**, 064003 (2006)
22. Jiang, Q.Q., Wu, S.Q., Cai, X.: Phys. Lett. B **651**, 58 (2007)
23. Jiang, Q.Q., Wu, S.Q., Cai, X.: Phys. Lett. B **651**, 65 (2007)
24. Jiang, Q.Q.: Class. Quantum Gravity **24**, 4391 (2007)
25. Jiang, Q.Q., Wu, S.Q., Cai, X.: Phys. Lett. B **647**, 200 (2007)
26. Jiang, Q.Q.: Phys. Rev. D **78**, 044009 (2008)
27. Jiang, Q.Q.: Phys. Lett. B **666**, 517 (2008)
28. Chen, D.Y., Jiang, Q.Q., Zu, X.T.: Phys. Lett. B **665**, 106 (2008)
29. Chen, D.Y., Jiang, Q.Q., Zu, X.T.: Class. Quantum Gravity **25**, 205022 (2008)
30. Zhang, J.Y., Zhao, Z.: Phys. Lett. B **618**, 14 (2005)
31. Zhang, J.Y., Zhao, Z.: J. High Energy Phys. **0505**, 055 (2005)
32. Hu, Y.P., Zhang, J.J., Zhao, Z.: Mod. Phys. Lett. A **21**, 2143 (2006)
33. Liu, C.Z., Zhang, J.Y., Zhao, Z.: Phys. Lett. B **639**, 670 (2006)
34. Liu, W.B.: Phys. Lett. B **634**, 541 (2006)
35. Bekenstein, J.D.: Phys. Rev. D **7**, 2333 (1973)
36. t'Hooft, G.: Nucl. Phys. B **256**, 727 (1985)
37. Jing, J.L.: Int. J. Theor. Phys. **37**, 1441 (1998)
38. Lee, M.H., Kim, J.K.: Phys. Lett. A **212**, 323 (1996)
39. Ghosh, A., Mitra, P.: Phys. Lett. B **357**, 295 (1995)
40. Demer, J., Lafrande, R., Myers, C.R.: Phys. Rev. D **52**, 2245 (1995)
41. Li, X., Zhao, Z.: Phys. Rev. D **62**, 104001 (2000)
42. Liu, W.B., Zhu, J.Y., Zhao, Z.: Acta Phys. Sin. **49**, 581 (2000)
43. Gao, C.J., Shen, Y.G.: Chin. Phys. Lett. **18**, 1167 (2001)
44. He, H., Zhao, Z.: Acta Phys. Sin. **51**, 2661 (2002)
45. Zhu, B., Yao, G.Z., Zhao, Z.: Acta Phys. Sin. **51**, 2656 (2002)
46. Meng, Q.M., Su, J.Q., Li, C.A.: Acta Phys. Sin. **52**, 1822 (2003)
47. Han, Y.W., Hong, Y., Yang, S.Z.: Acta Phys. Sin. **56**, 10 (2007)
48. Han, Y.W., Hong, Y.: Acta Phys. Sin. **53**, 3270 (2004)
49. Han, Y.W., Yang, S.Z., Liu, W.B.: Commun. Theor. Phys. **43**, 382 (2005)
50. Liu, W.B., Han, Y.W., Zhao, Z.: Int. J. Mod. Phys. A **18**, 2681 (2003)
51. Liu, Z.F., Li, X., Zhao, Z.: J. Beijing Norm. University **39**, 751 (2003)
52. Sun, X.F., Jing, J.L., Liu, W.B.: Acta Phys. Sin. **53**, 4002 (2004)
53. Li, X.: Phys. Lett. B **540**, 9 (2002)
54. Yang, S.Z.: Acta Phys. Sin. **53**, 4007 (2004)
55. Liu, Z.F., Liu, W.B.: Acta Phys. Sin. **54**, 475 (2005)
56. Jiang, Q.Q., Yang, S.Z., Li, H.L.: Chin. Phys. **14**, 1736 (2005)
57. Li, X., Zhao, Z.: Mod. Phys. Lett. A **15**, 1739 (2000)